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Universal Z_N-graded differential calculus

Michel Dubois-Violette [a,*], Richard Kerner [b]

[a] *Laboratoire de Physique Théorique et Hautes Energies, Université Paris-XI, CNRS – URA D0 063, Université Paris-Sud, Bâtiment 211, 91405 Orsay, France*

[b] *Laboratoire de Gravitation et Cosmologie Relativistes, Université Pierre-et-Marie-Curie, CNRS – URA D0 769, Tour 22, 4-ème étage, Boîte 142, 4, Place Jussieu, 75005 Paris, France*

Abstract

We investigate the properties of differential algebras generated by an operator d satisfying the property $\mathrm{d}^N = 0$ instead of $\mathrm{d}^2 = 0$ as in the usual case. Several examples of realizations of such differential algebras are given, either in the context of Z_N-graded $N \times N$ matrix algebras, or as a generalized differential calculus on manifolds.

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1. Introduction

In several recently published papers [1–4,7] the Z_3-graded algebras have been investigated. Some of them were associative, like the Z_3-graded generalizations of Grassmann and Clifford algebras, some of them were not, like the algebra of cubic matrices [3], which is a *ternary* algebra, with the internal composition $m : \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ which in general cannot be induced by a *binary* composition law. In [5,6] a more general notion of *n-algebras* has been introduced, investigated, and some applications in mathematical physics have been suggested. Finally, in [9], a formal differential calculus with exterior derivation d satisfying $\mathrm{d}^3 = 0$ has been introduced and a particular model has been investigated. It leads to an interesting version of a gauge theory with a 3-form replacing the usual curvature represented by a 2-form, and to higher-order differential equations for the gauge fields.

It turns out that such a scheme can be easily extended to a more general case when one postulates $\mathrm{d}^N = 0$. The differential algebra generated by the entities $\mathrm{d}x^k$, d^2x^k, d^3x^k, $\ldots$ $\mathrm{d}^{(N-1)}x^k$ has a natural Z_N-grading [10].

* Corresponding author.

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By imposing some consistent (but by no means unique) cyclic commutation relations on the expressions of order N it can be made finite. In particular, we discuss the realization in which the operator d satisfies a q-graded Leibniz rule:

$$\mathrm{d}(\omega\phi) = (\mathrm{d}\omega)\phi + q^{|\omega|}\omega(\mathrm{d}\phi),$$

where q is a primitive Nth root of unity, i.e. $q^N = 1$, but $q^k \neq 1$ for $k = 1, 2, \ldots, N-1$. The products of any two forms resulting in a form of highest order (N) (i.e. with $\deg(\omega) = p$, and $\deg(\phi) = (N-p)$, so that $\omega\phi$ is of Z_N-degree $N = 0_{\mathrm{mod}(N)}$) satisfy the following commutation relations:

$$\omega\phi = q^{|\omega||\phi|}\phi\omega = q^{-|\omega|^2}\phi\omega.$$

It is also possible to generalize the notions of *p-cycles* and *p-boundaries*, and to compute their generalized cohomologies. Also a generalized version of Stokes' theorem can be put forward in quite an obvious way, namely, for a 1-form ω one has

$$\int_{\partial^{N-1}C} \omega = \int_{\partial^{N-2}C} \mathrm{d}\omega \cdots = \int_{\partial C} \mathrm{d}^{N-2}\omega = \int_C \mathrm{d}^{N-1}\omega. \tag{1}$$

In what follows, we show how such differential calculus can be realized on complex matrix algebras, then on differential manifolds; finally, we give the construction of universal model of such differential calculus based on the tensor powers of given unital algebra.

2. Algebraic differential calculus of higher-order

Consider the algebra $\mathcal{A} = \mathrm{Mat}_3(\mathbb{C})$ of 3×3 complex matrices. It can be naturally represented as a direct sum of three linear subspaces, $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1 \oplus \mathcal{A}_2$, defined as follows:

$$\mathcal{A}_0\colon \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}, \qquad \mathcal{A}_1\colon \begin{pmatrix} 0 & \alpha & 0 \\ 0 & 0 & \beta \\ \gamma & 0 & 0 \end{pmatrix}, \qquad \mathcal{A}_2\colon \begin{pmatrix} 0 & 0 & \gamma \\ \alpha & 0 & 0 \\ 0 & \beta & 0 \end{pmatrix}. \tag{2}$$

Arbitrary matrices belonging to $\mathcal{A}_k$, $k = 0, 1, 2$, are said to have respective degree 0, 1 and 2. It is easy to check that these degrees add up mod 3 under the associative matrix multiplication law.

Let B, C denote two matrices whose degree are b and c, respectively. We can define the Z_3- *graded commutator* $[B, C]$ as follows:

$$[B, C]_{Z_3} := BC - j^{bc}CB, \tag{3}$$

where $j = \mathrm{e}^{2\pi \mathrm{i}/3}$, $j^2 = \mathrm{e}^{4\pi \mathrm{i}/3}$, $j^3 = 1$, $1 + j + j^2 = 0$ (note that this Z_3-graded commutator does not satisfy the Jacobi identity). Let η be a matrix of degree 1; we can choose for the sake of simplicity

$$\eta = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \tag{4}$$

With the help of the matrix η we can define a formal "differential" on the Z_3-graded algebra of 3 matrices as follows:

$$\mathrm{d}B := [\eta, B]_{Z_3} = \eta B - j^b B\eta. \tag{5}$$

It is easy to show that $\mathrm{d}(BC) = (\mathrm{d}B)C + j^b B(\mathrm{d}C)$ and that $\mathrm{d}^3 = 0$. The first identity is trivial, whereas the last one follows from the fact that $\eta^3 = I\mathrm{d}$ does commute with all the elements of the algebra:

$$\begin{aligned} \mathrm{d}^3 B &= [\eta, [\eta, [\eta, B]_{Z_3}]_{Z_3}]_{Z_3} = [\eta, [\eta, (\eta B - j^b B\eta)]_{Z_3}]_{Z_3} = \cdots \\ &= j^b(1 + j + j^2)[\cdots] + \eta^3 B - B\eta^3 = 0 \end{aligned} \tag{6}$$

(because $\eta^3 \sim \mathbf{1}$, and commutes with all the elements of $\mathcal{A}$).

A similar construction can be performed in the case of $n \otimes n$ complex matrices, with q being a primitive nth root of unity, $q^n = 1$. Such an algebra is naturally Z_n-graded, with *diagonal matrices* representing degree 0, and degree 1 elements represented by the matrices whose only n non-vanishing entries are placed directly above the main diagonal $(n - 1)$ elements, the last one (the nth) placed in the lowest left case. This gives, for $n = 4$, four sets of matrices generated by the consecutive powers of the following matrix M:

$$M = \begin{pmatrix} 0 & \alpha & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \gamma \\ \delta & 0 & 0 & 0 \end{pmatrix} \in \mathcal{A}_1, \quad M^2 \in \mathcal{A}_2, \quad M^3 \in \mathcal{A}_3, \qquad M^4 \in \mathcal{A}_0. \tag{7}$$

M^4 spans the set of diagonal matrices to which we attribute degree 0. The degrees 0, 1, 2 and 3 add mod 4 under matrix multiplication. Now, a graded q-derivation of degree 1 can be defined as

$$\mathrm{Der}_A(B) = [A, B]_{Z_n} = AB - (q)^{\deg(B)} BA$$

with A an arbitrary degree 1 matrix; we can choose $q = \mathrm{i} = \mathrm{e}^{\mathrm{i}\pi/2}$.

The same definition can be written as

$$\mathrm{ad}_q(A)(B) = AB - q^{\deg(B)} BA.$$

Let us identify the matrices of Z_3-degree 0, 1 and 2 as the 0-forms, 1-forms and 2-forms, respectively, and their exterior Z_3-graded differentials as the Z_3-graded commutators with the matrix η, e.g. with

$$\eta = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \omega = \begin{pmatrix} 0 & \alpha & 0 \\ 0 & 0 & \beta \\ \gamma & 0 & 0 \end{pmatrix},$$

we have

$$\mathrm{d}\omega = [\eta, \omega]_{Z_3} = \eta\omega - j\omega\eta = \begin{pmatrix} 0 & 0 & (\beta - j\alpha) \\ (\gamma - j\beta) & 0 & 0 \\ 0 & (\alpha - j\gamma) & 0 \end{pmatrix}$$

so that here ω is of degree 1 and $\mathrm{d}\omega$ is of degree 2.

In the space of complex 3×3 matrices we can represent not only the p-forms (with $p = 0$, 1 and 2), but also *chains* over which these forms can be formally integrated. We define the p-chains as matrices whose degree is $(3 - p)_{\mathrm{mod}(3)}$, e.g. a chain C of degree 2 is given by a matrix belonging to $\mathcal{A}_1$.

We shall define the operation of *taking the border* of a chain as the Z_3-graded commutator of the corresponding matrix with the matrix η^T whose degree is 2:

$$\partial C = [\eta^{\mathrm{T}}, C] = \eta^{\mathrm{T}} C - j^{3-\deg(C)} C \eta^{\mathrm{T}}. \tag{8}$$

It is easy to see that $\partial^3 C = 0$ for any C, using the fact that $(\eta^{\mathrm{T}})^3 = \mathbf{1}$. Defining the integral of a p-form over a p-chain C as the trace of the matrix $C^{\mathrm{T}}\omega$:

$$\int_C \omega = \langle C, \omega\rangle = \mathrm{Tr}(C^{\mathrm{T}}\omega), \tag{9}$$

one easily proves the following generalization of Stokes' formula:

$$\langle \partial^2 C, \omega\rangle = \langle \partial C,\ \mathrm{d}\omega\rangle = \langle C,\ \mathrm{d}^2\omega\rangle. \tag{10}$$

For, take for example $\int_C \mathrm{d}\omega$, which is by definition

$$\int_C \mathrm{d}\omega = \mathrm{Tr}[C^{\mathrm{T}}\,\mathrm{d}\omega] = \mathrm{Tr}[C^{\mathrm{T}}(\eta\omega - j^{|\omega|}\omega\eta)] = \mathrm{Tr}[C^{\mathrm{T}}\eta\omega - j^{|\omega|}C^{\mathrm{T}}\omega\eta]. \tag{11}$$

Let us compute $\langle \partial C \omega\rangle$: according to the definition,

$$\begin{aligned}\langle \partial C\omega\rangle &= \mathrm{Tr}[(\partial C)^{\mathrm{T}}\omega] = \mathrm{Tr}[(\eta^{\mathrm{T}}C - j^{3-|C|}C\eta^{\mathrm{T}})^{\mathrm{T}}\omega] \\ &= \mathrm{Tr}[(C^{\mathrm{T}}\eta - j^{2.|C|}\eta C^{\mathrm{T}})\omega] = \mathrm{Tr}[C^{\mathrm{T}}\eta\omega - j^{2.|C|}\eta C^{\mathrm{T}}\omega],\end{aligned} \tag{12}$$

where we use the shortened notation $|\omega| = \deg(\omega)$ and $|C| = \deg(C)$.

Now, the first term is exactly as in the previous formula, whereas the second term is equal to $-j^{3-|C|}\mathrm{Tr}[C^{\mathrm{T}}\omega\eta]$ because the trace of a product of any number of matrices is invariant under a cyclic permutation; therefore, the second term will be equal to the second term of the previous formula, if $|C|) + |\omega| = 3$, which we assumed in our definition.

The same scheme can be used for any higher grading, e.g. in the case of the Z_4-graded algebra of 4×4 matrices introduced in the beginning of this Section, we may choose any degree 1 matrix as η.

Then the matrix η^{T} is of degree 3, and in order for our generalization of Stokes' formula to hold, we should define a p-chain as a matrix C whose degree is $(4 - p)_{\mathrm{mod}(4)}$, and in a more general case of Z_N-graded matrix differential algebra, as a $(N - p)_{\mathrm{mod\,N}}$-degree matrix.

It is not difficult to find in each of the components of the Z_3-graded 3×3 complex matrices the subspaces defined as Ker(d), Ker(d^2), Im(d) and Im(d^2), with usual inclusions:

$$\mathrm{Im}(\mathrm{d}) \subset \mathrm{Ker}(\mathrm{d}^2), \qquad \mathrm{Im}(\mathrm{d}^2) \subset \mathrm{Ker}(\mathrm{d}),$$
$$\mathrm{Im}(\mathrm{d}^2) \subset \mathrm{Im}(\mathrm{d}), \qquad \mathrm{Ker}(\mathrm{d}) \subset \mathrm{Ker}(\mathrm{d}^2).$$

A 0-form must be represented by a 0-degree (diagonal) 3×3 matrix:

$$f = \begin{pmatrix} f_1 & 0 & 0 \\ 0 & f_2 & 0 \\ 0 & 0 & f_3 \end{pmatrix},$$

whose differential is

$$\mathrm{d}f = \eta f - f\eta = \begin{pmatrix} 0 & f_2 - f_1 & 0 \\ 0 & 0 & f_3 - f_2 \\ f_1 - f_3 & 0 & 0 \end{pmatrix}$$

so that the condition for $\mathrm{d}f = 0$ amounts to $f_1 = f_2 = f_3$.

The second differential of f, $\mathrm{d}^2 f$, is equal to

$$\mathrm{d}^2 f = \eta\,\mathrm{d}f - j\,\mathrm{d}f\eta = (f_1 + jf_2 + j^2 f_3)\begin{pmatrix} 0 & 0 & j \\ 1 & 0 & 0 \\ 0 & j^2 & 0 \end{pmatrix}$$

so that the condition $\mathrm{d}^2 f = 0$ is equivalent with $f_1 + jf_2 + j^2 f_3 = 0$. This equation has *two* independent solutions:

$$f_1 = f_2 = f_3 \quad \text{and} \quad f_1 = j^2 f_2 = jf_3.$$

The first solution implies $\mathrm{d}f = 0$, and a fortiori $d^2 f = 0$, whereas the second implies $\mathrm{d}^2 f = 0$ but $\mathrm{d}f \neq 0$. Therefore, in the 0-degree sector, we have Ker(d) $\subset$ Ker(d^2), but Ker(d) $\neq$ Ker(d^2) which is true also for the other two sectors.

A similar situation is observed in the sector of degree 1 (the 1-forms); the identification of matrices representing Ker(d), Ker(d^2), Im(d), etc. in the case of 1-, 2- and 3-forms is a simple exercise.

At the end, one can see that the total space of p-forms covers the entire space of complex 3×3 matrices:

$$\dim(\Lambda) = \dim(\Lambda^0) + \dim(\Lambda^1) + \dim(\Lambda^2) = 3 + 3 + 3 = 9$$

with the following structure with respect to the operators d and d^2:

$$\mathrm{Ker}(\mathrm{d}) = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & j & 0 \\ 0 & 0 & j^2 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ j^2 & 0 & 0 \\ 0 & j & 0 \end{pmatrix} \right\},$$

$$\mathrm{Ker}(\mathrm{d}^2) = \mathrm{Ker}(\mathrm{d}) \oplus \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & j & 0 \\ 0 & 0 & j^2 \end{pmatrix}, \begin{pmatrix} 0 & j^2 & 0 \\ 0 & 0 & j \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \right\}.$$

It is also easy to show that one has in this case

$$\mathrm{Im}(\mathrm{d}^2) = \mathrm{Ker}(\mathrm{d}) \quad \text{and} \quad \mathrm{Im}(\mathrm{d}) = \mathrm{Ker}(\mathrm{d}^2).$$

More generally, if E is a vector space equipped with an endomorphism d satisfying $\mathrm{d}^N = 0$, one has $\mathrm{Im}(\mathrm{d}^{N-n}) \subset \mathrm{Ker}(\mathrm{d}^n)$ so one can introduce the vector spaces $H^{(n)} = \mathrm{Ker}(\mathrm{d}^n)/\mathrm{Im}(\mathrm{d}^{N-n})$ for $n = 1, \ldots, N-1$. The vector spaces $H^{(n)}$ generalize the homology; they are not independent and one can show (see [11]) that they are connected by a finite family of exact hexagons of homomorphisms for $N \geq 3$.

A formal algebraic analogue of connection and curvature forms have been discussed elsewhere (see [9]). In the next section we shall show how such Z_n-graded exterior calculus may be realized on a differential manifold.

3. N-graded differential calculus on a manifold

Let us show how a graded differential calculus with $\mathrm{d}^N = 0$ can be defined and developed on a manifold, generalizing the usual exterior differential by replacing -1 by q, a primitive Nth root of unity ($N \geq 2$).

Let M be a C^∞ differential manifold of dimension D, and let $\mathcal{F}(M)$ be the algebra of C^∞ functions over M. The operator d maps $\mathcal{F}(M)$ into the linear space Λ^1 of 1-forms which is a left module over the algebra $\mathcal{F}(M)$. As in the usual case, we suppose that Λ^1 is spanned locally by D 1-forms $\mathrm{d}\xi^k$ which are the first differentials of local coordinates ξ^k, $k = 1, 2, \ldots, D$, which belong to $\mathcal{F}(M)$. Now, in the usual Z_2-graded case one has $\mathrm{d}^2\xi^k = 0$. Because this fact should be independent of the choice of local coordinate system, d^2 should vanish when applied to any function of the coordinates ξ^k.

If we introduce the *q-graded Leibniz rule* as usual, by postulating the existence of an associative product for the elements of Λ, and setting

$$\mathrm{d}(\omega\phi) = (\mathrm{d}\omega)\phi + q^{|\omega|}\omega(\mathrm{d}\phi),$$

then for a function f we shall define

$$\mathrm{d}f(\xi^m) = \frac{\partial f}{\partial \xi^k}\,\mathrm{d}\xi^k \in \Lambda^1.$$

In the usual Z_2-graded case we require that $\mathrm{d}^2 f = 0$. This leads to the following equation:

$$\mathrm{d}^2 f = \frac{\partial^2 f}{\partial \xi^k \partial \xi^l}\,\mathrm{d}\xi^k\,\mathrm{d}\xi^l + \frac{\partial f}{\partial \xi^l}\,\mathrm{d}^2\xi^l = 0. \tag{13}$$

The second term vanishes by virtue of the fact that $\mathrm{d}^2\xi^l = 0$ by definition; therefore the first one has to vanish always, too. This is achieved by stating that the associative product of 1-forms $\mathrm{d}\xi^k\,\mathrm{d}\xi^l$ must be *antisymmetric*, i.e. that we have

$$\mathrm{d}\xi^k\,\mathrm{d}\xi^l = -\,\mathrm{d}\xi^l\,\mathrm{d}\xi^k.$$

(To underline this fact one usually denotes the so defined "exterior" product by inserting a wedge sign, $\mathrm{d}\xi^k \wedge \mathrm{d}\xi^l = -\mathrm{d}\xi^l \wedge \mathrm{d}\xi^k$.)

Now, let q be a primitive Nth root of unity, $q^N = 1$, but $q \neq 1$. If we impose on the operator d the q-graded Leibniz rule as above, and if we require that $\mathrm{d}^N = 0$, we can impose consistently the following minimal set of generalized commutation rules on the products of forms of order N:

$$\mathrm{d}\xi^{k_1}\,\mathrm{d}\xi^{k_2}\cdots\mathrm{d}\xi^{k_N} = q\,\mathrm{d}\xi^{k_2}\cdots\mathrm{d}\xi^{k_N}\,\mathrm{d}\xi^{k_1} = q^2\,\mathrm{d}\xi^{k_3}\cdots\mathrm{d}\xi^{k_N}\,\mathrm{d}\xi^{k_1}\,\mathrm{d}\xi^{k_2} = \cdots \tag{14}$$

As a corollary, one can conjecture that for $N \geq 3$ any product of more than N such 1-forms must vanish. For small values of $N (\leq 20)$ this can be easily seen by performing several consecutive permutations and using the associativity of the product of forms. For example, for $N = 3$,

$$\begin{aligned}\mathrm{d}\xi^i\,\mathrm{d}\xi^k\,\mathrm{d}\xi^l\,\mathrm{d}\xi^m &= j\,\mathrm{d}\xi^k\,\mathrm{d}\xi^l\,\mathrm{d}\xi^i\,\mathrm{d}\xi^m = j^2\,\mathrm{d}\xi^k\,\mathrm{d}\xi^i\,\mathrm{d}\xi^m\,\mathrm{d}\xi^l\\ &= j^3\,\mathrm{d}\xi^i\,\mathrm{d}\xi^m\,\mathrm{d}\xi^k\,\mathrm{d}\xi^l = j^4\,\mathrm{d}\xi^i\,\mathrm{d}\xi^k\,\mathrm{d}\xi^l\,\mathrm{d}\xi^m\\ &= j\,\mathrm{d}\xi^i\,\mathrm{d}\xi^k\,\mathrm{d}\xi^l\,\mathrm{d}\xi^m\end{aligned}$$

and because $j^4 = j \neq 1$, the whole expression must vanish.

As now $\mathrm{d}^2 \neq 0$, $\mathrm{d}^3 \neq 0, \ldots, \mathrm{d}^{N-1} \neq 0$, we must introduce new independent differentials

$$\mathrm{d}^2\xi^k,\ \mathrm{d}^3\xi^k, \ldots,\ \mathrm{d}^{N-1}\xi^k.$$

Each kind of these new "*1-forms of degree m*" with $m = 1, 2 \ldots (N-1)$ spans a basis of a D-dimensional linear space.

We shall assume that all the products of forms whose total degree is less than N are independent and span new modules over the algebra of functions with appropriate dimensions, e.g. the products of degree 2, $\mathrm{d}\xi^k\,\mathrm{d}\xi^m$, span a D^2-dimensional linear space; so do the products $\mathrm{d}^2\xi^k\,\mathrm{d}\xi^m$ and, independently, $\mathrm{d}\xi^m\,\mathrm{d}^2\xi^k$ (if $D > 3$), and so on. On the other hand, all other products of degree N must obey the following commutation relations, which are compatible with the cyclic commutation relations for the product of N 1-forms, for example:

$$\begin{aligned}&\mathrm{d}^p\xi^k\,\mathrm{d}^{N-p}\xi^l = q^p\,\mathrm{d}^{N-p}\xi^l\,\mathrm{d}^p\xi^k,\\ &\mathrm{d}^{N-p}\xi^k\,\mathrm{d}\xi^{l_1}\,\mathrm{d}\xi^{l_2}\cdots\mathrm{d}\xi^{l_p} = q^{N-p}\,\mathrm{d}\xi^{l_1}\,\mathrm{d}\xi^{l_2}\cdots\mathrm{d}\xi^{l-p}\,\mathrm{d}^{N-p}\xi^k,\end{aligned} \tag{15}$$

and so on.

Finally, we shall assume that not only the products of $N+1$ and more 1-forms vanish, but along with them, also any other products of all kinds of forms whose total degree is greater than N. This additional assumption is necessary in order to ensure the coordinate-independent character of the condition $\mathrm{d}^N = 0$. As a matter of fact, under a coordinate change all the products of forms of given order mix up and transform into each other, e.g. the terms like $\mathrm{d}\xi^j\,\mathrm{d}\xi^k\,\mathrm{d}\xi^l$ with the terms of the type $\mathrm{d}^2\xi^k\,\mathrm{d}\xi^l$, and similarly for higher-order terms.

Let us show now the explicit expressions for $\mathrm{d}^p f(\xi^k)$. Using the rules introduced above, we have

$$\begin{aligned}
\mathrm{d}^2 f &= \frac{\partial^2 f}{\partial\xi^k\partial\xi^l}\,\mathrm{d}\xi^k\,\mathrm{d}\xi^l + \frac{\partial f}{\partial\xi^k}\,\mathrm{d}^2\xi^k,\\
\mathrm{d}^3 f &= \frac{\partial^3 f}{\partial\xi^k\partial\xi^l\partial\xi^m}\,\mathrm{d}\xi^k\,\mathrm{d}\xi^l\,\mathrm{d}\xi^m + \frac{\partial^2 f}{\partial\xi^k\partial\xi^l}(\mathrm{d}^2\xi^k\,\mathrm{d}\xi^l + q\,\mathrm{d}\xi^k\,\mathrm{d}^2\xi^l)\\
&\quad + \frac{\partial^2 f}{\partial\xi^l\partial\xi^k}\,\mathrm{d}\xi^l\,\mathrm{d}^2\xi^k + \frac{\partial f}{\partial\xi^k}\,\mathrm{d}^3\xi^k,\\
\mathrm{d}^4 f &= \partial^4_{klmn} f\,\mathrm{d}\xi^k\,\mathrm{d}\xi^l\,\mathrm{d}\xi^m\,\mathrm{d}\xi^n\\
&\quad + \partial^3_{klm} f[\mathrm{d}^2\xi^k\,\mathrm{d}\xi^l\,\mathrm{d}\xi^m + (1+q)\,\mathrm{d}\xi^k\,\mathrm{d}^2\xi^l\,\mathrm{d}\xi^m\\
&\qquad + (1+q+q^2)\,\mathrm{d}\xi^k\,\mathrm{d}\xi^l\,\mathrm{d}^2\xi^m]\\
&\quad + \partial^2_{kl} f[\mathrm{d}^3\xi^k\,\mathrm{d}\xi^l + (1+q+q^2)\,\mathrm{d}\xi^k\,\mathrm{d}^3\xi^l\\
&\qquad + (1+q+q^2)\,\mathrm{d}^2\xi^k\,\mathrm{d}^2\xi^l] + \partial_k f\,\mathrm{d}^4\xi^k
\end{aligned}$$

and so forth.

It is easy to prove that for a given N it is enough to assume $\mathrm{d}^N\xi^k = 0$ and the N-cyclic commutation rule

$$\mathrm{d}\xi^{k_1}\,\mathrm{d}\xi^{k_2}\,\mathrm{d}\xi^{k_3}\cdots\mathrm{d}\xi^{k_N} = q\,\mathrm{d}\xi^{k_2}\,\mathrm{d}\xi^{k_3}\cdots\mathrm{d}\xi^{k_N}\,\mathrm{d}\xi^{k_1}$$

implemented with its generalization for any product of two exterior forms of the total order adding up to N,

$$\omega\phi = q^{p(N-p)}\phi\omega = q^{-p^2}\phi\omega$$

whenever $\deg(\omega) = p$ and $\deg(\phi) = N - p$, in order to ensure that $\mathrm{d}^N f = 0$, and in general, $\mathrm{d}^N\omega = 0$ for any differential form ω.

The dimension of an N-graded differential algebra with D generators $\mathrm{d}\xi^k$ $(k, l \ldots = 1, 2, \ldots, D)$ cannot be given by any simple and concise formula, because it depends crucially on whether N is a prime number or not. But it is easy to determine this dimension for the first few cases, $N = 3, 4, 5$.

For example, in the Z_3-graded case it is easy to check that taking into account the commutation relations that hold for the products of forms with total degree equal to 3, we have the following basis in the space of forms:

$$1\ \text{0-form}(\mathbf{1}),\quad D\ \text{1-forms}\,\mathrm{d}\xi^k,\quad D^2\ \text{2-forms}\,\mathrm{d}\xi^k\,\mathrm{d}\xi^m,$$

besides, we have also

$$\tfrac{1}{3}(D^3 - D)\ \text{3-forms}\,\mathrm{d}\xi^k\,\mathrm{d}\xi^l\,\mathrm{d}\xi^m,\quad D\ \text{forms}\,\mathrm{d}^2\xi^k,\ \text{and}\ D^2\ \text{forms}\ \mathrm{d}^2\xi^k\,\mathrm{d}\xi^m,$$

which gives the total dimension of the algebra $= 1 + 2D + 2D^2 + \frac{1}{3}(D^3 - D)$.

The case of $N = 4$ is more complicated, because 4 is not a prime number. Here the cyclic q-commutation relation:

$$\mathrm{d}\xi^k\,\mathrm{d}\xi^l\,\mathrm{d}\xi^m\,\mathrm{d}\xi^n = i\,\mathrm{d}\xi^l\,\mathrm{d}\xi^m\,\mathrm{d}\xi^n\,\mathrm{d}\xi^k = -\,\mathrm{d}\xi^m\,\mathrm{d}\xi^n\,\mathrm{d}\xi^k\,\mathrm{d}\xi^l$$

implies also the *anticommutation* rule for the couples $\mathrm{d}\xi^k\,\mathrm{d}\xi^m$. Therefore, the dimension of the degree 4 products of the $\mathrm{d}\xi^m$ is now $\frac{1}{4}[D^4 - D - D(D-1)] = \frac{1}{4}(D^4 - D^2)$, because now not only all the *D fourth powers* like $(\mathrm{d}\xi^k)^4$ must identically vanish, but also the $D(D-1)$ expressions of the type $(\mathrm{d}\xi^k\,\mathrm{d}\xi^l)(\mathrm{d}\xi^k\,\mathrm{d}\xi^l)$ (with $k \neq l$) must vanish, too, because in general

$$(\mathrm{d}\xi^k\,\mathrm{d}\xi^l)(\mathrm{d}\xi^m\,\mathrm{d}\xi^n) = -(\mathrm{d}\xi^m\,\mathrm{d}\xi^m)(\mathrm{d}\xi^k\,\mathrm{d}\xi^l).$$

The total dimension of the differential algebra generated by the forms

$$\mathrm{d}\xi^k,\quad \mathrm{d}^2\xi^l,\quad \mathrm{d}^3\xi^m$$

is therefore equal to $\frac{1}{4}(D^4 + 8D^3 + 17D^2 + 10D + 4)$.

For the case $N = 5(\mathrm{d}^5 = 0)$ the computation is again simpler, because 5 is a prime number, and the dimension of the fifth-order products of 1-forms is simply $\frac{1}{5}(D^5 - D) = \frac{1}{5}(D-1)D(D+1)(D^2+1)$.

The dimension of the subspace of Nth order products in the differential $\mathrm{d}^N = 0$ algebra spanned by D independent generators is given by the formula $(D^N - D)/N$ if N is a prime number; it is much more complicated if it is not.

4. Universal N-graded differential calculus

Our aim now is to construct the *universal differential calculus* for the higher-order differentials, the examples of which have been shown in the previous section.

Let $\mathcal{A}$ be an associative algebra with unit element and let $\Omega := \bigoplus_{k=0}^{\infty} \Omega^k$, be a graded associative algebra with $\Omega^0 = \mathcal{A}$; the elements of Ω^k are said to be of *degree k*.

A q-differential is a linear mapping of degree 1 of Ω into itself $\mathrm{d} : \Omega^k \longrightarrow \Omega^{k+1}$ such that if $\alpha \in \Omega^k$ and $\beta \in \Omega^m$, then

$$\mathrm{d}(\alpha\beta) = (\mathrm{d}\alpha)\beta + q^k\alpha(\mathrm{d}\beta) \quad \text{and} \quad \mathrm{d}^N = 0 \tag{16}$$

in the case when q is an Nth root of unity (i.e. $q^N = 1$).

A graded algebra equipped with a q-differential will be called a *graded q-differential algebra* or simply a *q-differential algebra* whenever no confusion arises and a q-differential algebra with $\mathcal{A} = \Omega^0$ as above will be refer red to as a *q-differential calculus* over $\mathcal{A}$.

Now we would like to define a *universal q-differential calculus* for the differentials of this sort. Let us notice that the q-differential induces a derivation of $\mathcal{A}$ into Ω^1. Let us recall in this context the construction of the universal derivation.

Let $\Omega^1_{\mathrm{u}}(\mathcal{A}) \subset \mathcal{A} \otimes \mathcal{A}$ be the kernel of the product $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ and let $\mathrm{d}_{\mathrm{u}} : \mathcal{A} \to \Omega^1_{\mathrm{u}}(\mathcal{A})$ be defined by:

$$\mathrm{d}_{\mathrm{u}} f = \mathbf{1} \otimes f - f \otimes \mathbf{1}.$$

It is clear that $\Omega^1_{\mathrm{u}}(\mathcal{A})$ is a bimodule over $\mathcal{A}$ and it is easy to check that $\mathrm{d_u}$ is a derivation.

The derivation $\mathrm{d_u} : \mathcal{A} \to \Omega^1_{\mathrm{u}}(\mathcal{A})$ satisfies the following universal property: for any bimodule Ω^1 over $\mathcal{A}$ and any derivation of $\mathcal{A}$ into Ω^1, there exists a *unique* homomorphism i_{d} of the bimodules $\Omega^1_{\mathrm{u}}(\mathcal{A})$ into Ω^1, such that $\mathrm{d} = i_{\mathrm{d}} \circ \mathrm{d_u}$. This universal property characterizes the pair $(\Omega^1_{\mathrm{u}}(\mathcal{A}), \mathrm{d_u})$ up to an isomorphism.

Let $\mathcal{T}^n(\mathcal{A}) = \mathcal{A}^{\otimes n+1}$ and $\mathcal{T}^0(\mathcal{A}) = \mathcal{A}$. In other words, $\mathcal{T}^n(\mathcal{A}) = \otimes^n_{\mathcal{A}}(\mathcal{A} \otimes \mathcal{A})$ so that $\mathcal{T}(\mathcal{A}) = \oplus_n \mathcal{T}^n(\mathcal{A})$ is the tensor algebra over $\mathcal{A}$ of the bimodule $\mathcal{A} \otimes \mathcal{A}$, with the obvious inclusion $\mathcal{T}^n(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{T}^m(\mathcal{A}) \subset \mathcal{T}^{n+m}(\mathcal{A})$.

Now we can introduce the q-derivation on this algebra as follows. Let

$$f_0 \otimes f_1 \otimes \cdots \otimes f_n \in \mathcal{T}^n(\mathcal{A}).$$

One has

$$f_0 \otimes f_1 \otimes \cdots \otimes f_n = f_0(\mathbf{1} \otimes \mathbf{1}) f_1 (\mathbf{1} \otimes \mathbf{1}) f_2 (\mathbf{1} \otimes \mathbf{1}) \cdots (\mathbf{1} \otimes \mathbf{1}) f_n,$$

because one has $f_0\, \mathbf{1} \otimes \mathbf{1} = f_0 \otimes \mathbf{1}$ and $(\mathbf{1} \otimes \mathbf{1}) f_1 = \mathbf{1} \otimes f_1$, etc.

For the action of d on $\mathcal{A}$ we choose the universal derivation combined with the inclusion $\Omega^1_{\mathrm{u}}(\mathcal{A}) \subset \mathcal{T}^1(\mathcal{A})$, i.e. $\mathrm{d}x = [1 \otimes 1, x]$ for $x \in \mathcal{A}$. We define d on $\mathbf{1} \otimes \mathbf{1} \in \mathcal{T}^1(\mathcal{A})$ by setting

$$\mathrm{d}(\mathbf{1} \otimes \mathbf{1}) = \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1}. \tag{17}$$

Now, as we can write $\mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} = (\mathbf{1} \otimes \mathbf{1})(\mathbf{1} \otimes \mathbf{1})$ then, denoting $\mathbf{1} \otimes \mathbf{1} = e$, we can write d being a q-derivation:

$$\mathrm{d}e = e^2, \qquad \mathrm{d}^2 e = \mathrm{d}e\, e + q\, e\, \mathrm{d}e = \mathrm{d}(e^2), \tag{18}$$

which implies that $\mathrm{d}e\, e = e\, \mathrm{d}e$ and we can write

$$\mathrm{d}^2 e = (1+q) e\, \mathrm{d}e = [2]_q e\, \mathrm{d}e.$$

Note that have we use the notation of "quantum integer", in which

$$[N]_q = 1 + q + q^2 + \cdots + q^{(N-1)}$$

Therefore, we can continue:

$$\begin{aligned} \mathrm{d}^3 e &= (1+q)\, \mathrm{d}(e^3) = (1+q)(1+q+q^2) e^2\, \mathrm{d}e \\ &= (1+q)(1+q+q^2) e^4 = [2]_q [3]_q e^4 \end{aligned}$$

and so on. By induction on N one has

$$\mathrm{d}^N e = [N!]_q e^{N-1}\, \mathrm{d}e = [N!]_q e^{N+1}, \qquad \mathrm{d}^N x = [N!]_q e^{N-1}\, \mathrm{d}x. \tag{19}$$

If $q^N = 1$, one has

$$[N]_q := 1 + q + q^2 + \cdots + q^{(N-1)} = 0,$$

which implies $[N!]_q = 0$ and therefore

$$\mathrm{d}^N e = 0, \qquad \mathrm{d}^N x = 0$$

The generalized Leibniz formula for the Nth differential can be written as

$$\mathrm{d}^N(\alpha\beta) = \sum_{k=0}^{N} \left[\frac{N}{k}\right]_q q^{|\alpha|(N-k)}\, \mathrm{d}^k(\alpha)\, \mathrm{d}^{N-k}(\beta).$$

where we use the notation

$$\left[\frac{N}{k}\right]_q = \frac{[N!]_q}{[k!]_q[(N-k)!]_q]}.$$

Note that if q is a primitive root of 1, then d^N is also a derivation (whereas other powers of d are not!), which implies $\mathrm{d}^N = 0$ since $\mathrm{d}^N = 0$ on the generators.

Let $\Omega_q(\mathcal{A})$ be the smallest q-differential subalgebra of $\mathcal{T}(\mathcal{A})$ which contains $\mathcal{A}$. This graded q-differential algebra [11] is characterized, up to an isomorphism, by the following universal property:

For any homomorphism Φ of unital algebra $\mathcal{A} \to \Omega^0$ where Ω^0 is the subalgebra of elements of degree zero of a q-differential algebra $\Omega = \otimes_{k=0}^{\infty}\Omega^k$ there is a unique homomorphism of q-differential algebras of $\Omega_q(\mathcal{A})$ into Ω which extends Φ.

This is why $\Omega_q(\mathcal{A})$ will be called a *universal q-differential calculus* over $\mathcal{A}$. For $N = 2$ ($q = -1$) this coincides [12] with the standard universal differential calculus over $\mathcal{A}$

It is worthwhile to note that there exists another possible definition of the universal differential of $\mathrm{d}e$, namely, instead of $\mathrm{d}e = e^2$ we may choose

$$\mathrm{d}'e = -qe^2$$

which implies

$$\mathrm{d}'^N e = [N!]_q(-q)^{N-1}\,\mathrm{d}'e\; e^{(N-1)} = [N!]_q(-q)^N e^{N+1} \tag{20}$$

and for $x \in \mathcal{A}$

$$\mathrm{d}'^N x = [N!]_q(-1)^{N-1}\,\mathrm{d}x\; e^{N-1} \tag{21}$$

and therefore again $\mathrm{d}'^N = 0$. However the q-differential subalgebra generated by $\mathcal{A}$ is still isomorphic to $\Omega_q(\mathcal{A})$.

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